Spin coherent states with monopole harmonics on the Riemann sphere for the Kravchuk oscillator

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Abstract

We consider a class of generalized spin coherent states by choosing the labeling coefficients to be monopole harmonics. The latters are L^2 eigenstates of the mth spherical Landau level on the Riemann sphere with $m \in \mathbb{Z}_+$. We verify that the Klauder's minimum properties for these states to be considered as coherent states are satisfied. We particularize them for the case of the Kravchuk oscillator and we obtain explicite expression for their wave functions. The associated coherent states transforms provide us with a Bargmann-type representation for the states of the oscillator Hilbert space. For the lowest level $m = 0$ indexing monopole harmonics, we identify the obtained coherent states to be those of Klauder-Perelomov type which were constructed in Ref. $[J.Math.Phys.48, 112106\,(2007)]$.

1 Introduction

The spin $SU(2)$ coherent states (SCS) were introduced in the early 1970's by Radcliffe [1], Gilmore [2] and Perelomov [3]. They are also named *atomic* or *Bloch* coherent states. This diversity of appellations reflects the range of domains in quantum physics where these objects play some role. One can introduce these states by following a probabilistic and Hilbertian scheme as explained in details in the book of Gazeau [4]. Precisely, the SCS are the field states that are superposition of the number states with appropriately chosen coefficients. These labelling coefficients are such that the associated photon-counting distribution is a binomial probability distribution [5]. In addition, these coefficients constitute an orthonormal basis of a Hilbert space of analytic functions on the Riemann sphere satisfying a certain growth condition.

Now, as in [6], we replace the usual labelling coefficients in the superposition defining the SCS by an orthonormal basis consisting of monopole harmonics that are L^2 eigenfunctions of an invariant Laplacian on the Riemann sphere corresponding to discrete eigenvalues (*spherical Landau levels*) to introduce a class of generalized spin coherent states (GSCSs). Each of these eigenvalue is of *finite* degeneracy. Here, we precisely verify that the basic minimum properties for the

constructed states to be considered as coherent states are satisfied. Namely, the conditions which have been formulated by Klauder [7] : (a) the continuity of labelling, (b) the fact that these states are normalizable but not orthogonal and (c) these states fulfilled the resolution of the identity with a positive weight function. Next, we particularize the GSCSs formalism for the case of the Hamiltonian of the Kravchuk oscillator [8] which is a *finite* model oscillator whose importance consists in the fact that it can be considered as a discrete analogue of the harmonic oscillator and that its eigenfunctions which are the kravchuk functions coincide with the harmonic oscillator functions in a certain limit. Next, we obtain explicitly the wave functions of these GSCS which enable us to write a Bargmann-type representation of any state in the oscillator Hilbert space. For the lowest level $m = 0$ indexing monopole harmonics, we identify the obtained coherent states to be those of Klauder-Perelomov type which were constructed in [9] .

The paper is organized as follows. Section 2 deals with some needed facts on the monopole harmonics. In Section 3, we discuss a class of generalized spin coherent states attached to monopole harmonics and verify that they satisfy the basic minimum properties of coherent states. In Section 4, we recall briefly the definition of the Kravchuk oscillator and its eigenstates. In section 5, we particularize the constructed coherent states for the case of the Kravchuk oscillator and we discuss a Bargmann-type representation for the states of the oscillator Hilbert space. In section 6, we focus on a particular case of the constructed coherent states, for which we establish a connection with known results.

2 Monopole harmonics on the Riemann sphere

We shall recall here some needed spectral properties the Dirac monopole Hamiltonian operator according to references $[10-11]$. For this, we start by identifying the sphere $\mathbb{S}^2$ with the extended complex plane $\mathbb{C}\cup\{\infty\} \equiv \overline{\mathbb{C}}$, called the Riemann sphere, via the stereographic coordinate $z = x + iy$, $x, y \in \mathbb{R}$. We shall work within a fixed coordinate neighborhood with coordinate z obtained by deleting the "point at infinity" $\{\infty\}$. Near this point we use instead of z the coordinate z^{-1}. In the stereographic coordinate z, the Hamiltonian operator of the Dirac monopole with charge $q = 2\nu$ in the notations of Veslov and Ferepontov ([10] , p.598) reads

$$H_{2\nu} = -\left(1+z\overline{z}\right)^2 \frac{\partial^2}{\partial z \partial \overline{z}} - \nu z \left(1+z\overline{z}\right) \frac{\partial}{\partial z} + \nu \overline{z} \left(1+z\overline{z}\right) \frac{\partial}{\partial \overline{z}} + \nu^2 \left(1+z\overline{z}\right) - \nu^2. \tag{2.1}$$

This operator acts on the sections of the $U(1)-$bundle with the first Chern class q. We have denoted by $\nu \geq 0$ the strength of the quantized magnetic field. The associated annihilation and creation operators are given by

$$A_\nu \quad : \quad = \left(1+|z|^2\right) \frac{\partial}{\partial \overline{z}} + \nu z; \tag{2.2}$$

$$A_\nu^* \quad : \quad = -\left(1+|z|^2\right) \frac{\partial}{\partial z} + (\nu+1)\, \overline{z} \tag{2.3}$$

The Hamiltonian $H_{2\nu}$ in (2.1) is acting in the Hilbert space $L^2\left(\mathbb{S}^2, d\mu\right)$ with $d\mu(z) = (1+z\overline{z})^{-2}\, d\eta(z)\,,\ d\eta(z) = \pi^{-1}dxdy$ being the Lebesgue measure on $\mathbb{C}$. To find the ground state, the authors in [10] solved the equation $A_\nu[\psi] = 0$ and found that ψ should be of the form

$$\psi(z) = (1+z\overline{z})^{-\nu}\,\phi(z) \tag{2.4}$$

where $\phi(z)$ is any holomophic function of z. Because of the condition $\psi \in L^2\left(\mathbb{S}^2\right)$, the function $\phi(z)$ must be a polynomial of order $\leq 2\nu$. This gives a space of dimension $2\nu+1$. Next, by making use of the operator A_ν^* they have obtained that the eigenfunctions of the Dirac monopole $H_{2\nu}$ corresponding to the eigenvalue

$$\lambda_{\nu,m} := (2m+1)\,\nu + m\,(m+1)\,,\ m = 0,1,2,... \tag{2.5}$$

form a space of dimension $2\nu+2m+1$ and are described as

$$\Phi(z) = A_{\nu-m}^* ... A_{\nu-2}^* A_{\nu-1}^* \,(1+z\overline{z})^{-\nu-m}\, P(z)\,. \tag{2.6}$$

and $P(z)$ is a polynomial of degree $\leq 2\nu+2m$. The eigenfunctions of the Dirac monopole are known as monopole harmonics and have been investigated by Wu and Yang [12], who were probably the first to identify them as sections. To obtain explicit expressions of these eigen*sections* in the coordinate z, one also can make use of the results obtained by Peetre and Zhang [13] first by establishing an intertwining relation between the shifted operator

$$H_{2\nu} - \nu = (1+z\overline{z})^{-\nu}\,\Delta_{2\nu}\,(1+z\overline{z})^{\nu}\,, \tag{2.7}$$

and an invariant Laplacian

$$\Delta_{2\nu} := -\,(1+z\overline{z})^2\,\frac{\partial^2}{\partial z \partial \overline{z}} + 2\nu\overline{z}\,(1+z\overline{z})\,\frac{\partial}{\partial \overline{z}} \tag{2.8}$$

acting in the Hilbert space $L^{2,\nu}\left(\mathbb{S}^2\right) := L^2\left(\mathbb{S}^2, d\mu_\nu(z)\right)$ with $d\mu_\nu(z) = (1+z\overline{z})^{-2-2\nu}\, d\nu(z)\,.$ According to (2.7) any ket $\mid \phi >$ of $L^{2,\nu}\left(\mathbb{S}^2\right)$ is represented by

$$(1+z\overline{z})^{-\nu} < z \mid \phi > \text{ in } L^2\left(\mathbb{S}^2\right). \tag{2.9}$$

Next by ([13], *pp*.228 − 229), the relations (2.7), (2.9) and with the help of the polynomial functions

$$Q_j^{\nu,m}(u) = (j!)^{-1}\,(m+j)!\ {}_2F_1\left(-m, 2\nu+m+1, j+1; u\right) \tag{2.10}$$

where ${}_2F_1$ is the Gauss hypergeometric function [14], one can check that the functions

$$\Phi_j^{\nu,m}(z) := (1+z\overline{z})^{-\nu}\, z^j\ Q_j^{\nu,m}\left(\frac{z\overline{z}}{1+z\overline{z}}\right), \qquad -m \leq j \leq 2\nu+m \tag{2.11}$$

constitutes an orthogonal set in the eigenspace

$$\mathcal{A}_m^{\nu}\left(\mathbb{S}^2\right) = \left\{\Phi \in L^2\left(\mathbb{S}^2\right), H_{2\nu}\, \Phi = \lambda_{\nu,m}\, \Phi\right\} \tag{2.12}$$

corresponding to the eigenvalue $\lambda_{\nu,m}$ given in (2.5). But since the terminating ${}_2F_1$-sum can be written as

$${}_2F_1\left(k+\beta+\alpha+1, -k, 1+\alpha; \frac{1-t}{2}\right) = \frac{k!\Gamma\left(1+\alpha\right)}{\Gamma\left(k+1+\alpha\right)} P_k^{(\alpha,\beta)}\left(t\right) \tag{2.13}$$

in terms of the Jacobi polynomial ([14]) :

$$P_k^{(\alpha,\beta)}\left(t\right) = \sum_{s=0}^{k} \begin{pmatrix} k+\alpha \\ k-s \end{pmatrix} \begin{pmatrix} k+\beta \\ s \end{pmatrix} \left(\frac{t-1}{2}\right)^s \left(\frac{t+1}{2}\right)^{k-s} \tag{2.14}$$

then one can also present an orthonormal basis of the space (2.12) by the expression

$$\widetilde{\Phi}_j^{\nu,m}\left(z\right) := \gamma_j^{\nu,m}\left(1+z\overline{z}\right)^{-\nu} z^j P_m^{(j,2\nu-j)}\left(\frac{1-z\overline{z}}{1+z\overline{z}}\right) \tag{2.15}$$

where $-m \leq j \leq 2\nu+m$ and the constant is given by

$$\gamma_j^{\nu,m} := \sqrt{\frac{\left(2\nu+2m+1\right)\left(2\nu+m\right)!m!}{(m+j)!(2\nu+m-j)!}}. \tag{2.16}$$

We end this section by the following remarks.

Remark 3.1. In [10], the authors have also pointed out that if 2ν is even integer then $H_{2\nu}$ can be intertwined with the standard Laplace-Beltrami operator $-\Delta_{\mathbb{S}^2}$ on the sphere as follows:

$$H_{2\nu}D = D\left(-\Delta_{\mathbb{S}^2} - \nu^2\right), D = D_{\nu-1}D_{\nu-2}...D_1D_0, \tag{2.18}$$

where $D_\nu := \left(1+z\overline{z}\right)\overline{\partial} + \nu z$. So that one can use the eigenfunctions of $-\Delta_{\mathbb{S}^2}$, which are spherical harmonics, to construct the eigenfunctions of $H_{2\nu}$.

Remark 3.2. Note that the strength of the magnetic field ν should satisfy $\nu \in \left\{\frac{1}{2}, 1, \frac{3}{2}, ...\right\}$. As matter of fact, there is a result called Dirac's quantization for monopole charges which requires that the total flux of the magnetic field across a closed surface must be quantized, i.e. it must be an integer multiple of a universal constant. This result is about cohomology groups for hermitian line bundles [15] and is also known as the Weil-Souriau-Kostant quantization condition [16]. For more information on Dirac monopoles, see [17].

3 Generalized spin coherent states with monopole harmonics

Here, we shall make use of the same notation ν to consider a Hilbert space denoted $\mathcal{H}$ of dimension $(2\nu+1)$, carrying an irreducible representation of the

group $SU(2)$. Each space $\mathcal{H}$ is associated with a spin of length $\nu \in \left\{\frac{1}{2}, 1, \frac{3}{2}, ...\right\}$. To introduce spin-coherent states (SCS), it is convenient to select states of highest (lowest) weight $\mid \pm\nu >$ as reference states. These states are invariant under a change of phase, hence the isotopy group is given by $U(1)$. Therefore the closed space $SU(2)/U(1)$ is the surface of the sphere $\mathbb{S}^2$ identified with $\overline{\mathbb{C}}$ as mentioned above, which correspond to the phase space of a classical spin. The SCS defined in this Hilbert space are given by ([1] ,p.315):

$$\mid z, \nu, 0 >= (1 + z\overline{z})^{-\nu} \sum_{k=0}^{2\nu} \sqrt{\frac{\Gamma(2\nu+1)}{k!\Gamma(2\nu-k+1)}} \, |z|^k \, e^{ik\theta} \mid k, \nu > \tag{3.1}$$

where the labelling parameter $z = |z| e^{i\theta} \in \overline{\mathbb{C}}$ and $\mid k, \nu >$ are number states of the field mode. Due to the fact that the probability for the production of k photons, given by the quantity

$$|< k, \nu \mid z, \nu, 0 >|^2 = \frac{\Gamma(2\nu+1)}{k!\Gamma(2\nu-k+1)} (z\overline{z})^k (1 + z\overline{z})^{-2\nu} \tag{3.2}$$

is a binomial probability density $\mathfrak{B}(n, \tau)$ with $n := 2\nu$ as an integer parameter and $\tau := z\overline{z}(1 + z\overline{z})^{-1}$ as a Bernoulli parameter with $0 < \tau < 1$, the SCS in (3.1) is exactly of the form of the binomial state [18]. These states in (3.1) constitute a resolution of the identity of the Hilbert space

$$\int_{\mathbb{C}} \mid z, \nu, 0 >< 0, \nu, z \mid \frac{(2\nu+1)}{(1+z\overline{z})^2} = \mathbf{1}_{\mathcal{H}} \tag{3.3}$$

As pointed out in ([4], p.81) one should notice here the similarity with the standard coherent states

$$\mathbb{C} \ni \zeta \mapsto \mid \zeta >= \sum_{n=0}^{+\infty} \exp\left(-\frac{1}{2}\zeta\overline{\zeta}\right) \frac{\zeta^n}{\sqrt{n!}} \mid n > \tag{3.4}$$

which are obtained from the spin CS at the limit of high spin $N = 2\nu$ through a contraction process. The latter is carried out through a scaling of the complex variable z, namely $\zeta = \sqrt{N}z$ and $n = 2\nu - k$, $\mid k, \nu >\equiv \mid n >$ and the limit $N \to \infty$:

$$\mid z = \frac{\zeta}{\sqrt{N}} >_{spin} \to \mid \zeta > . \tag{3.5}$$

We also note that the coefficients in (3.1) can be written as

$$z \mapsto \sqrt{\frac{\Gamma(2\nu+1)}{j!\Gamma(2\nu-j+1)}} (1 + z\overline{z})^{-\nu} z^j = (2\nu+1)^{-\frac{1}{2}} \widetilde{\Phi}_j^{\nu,0}(z) \tag{3.6}$$

in terms of the elements $\widetilde{\Phi}_j^{\nu,0}(z)$ of the orthonormal basis of the space $\mathcal{A}_0^{\nu}(\mathbb{S}^2) = \left\{\Phi \in L^2(\mathbb{S}^2), H_{2s}\, \Phi = \lambda_{\nu,0}\Phi\right\}$ in (2.1) corresponding to the lowest energy level

$\lambda_{\nu,0}$. The latter is obtained by setting for $m=0$ in Eq.(2.5). From this observation, we propose as in [6] a generalization for these spin coherent states by following the probabilistic and Hilbertian scheme as explained in ([4], p.74). More precisely, we state the following.

Definition 3.1. *For each fixed* $m \in \mathbb{Z}_+$ *and* $\nu \in \left\{\frac{1}{2}, 1, \frac{3}{2}, ...\right\}$ *a class of generalized spin coherent state (GSCS) is defined by the form*

$$\mid z, \nu, m >:= \left(\mathcal{N}_{\nu,m}(z)\right)^{-\frac{1}{2}} \sum_{j=-m}^{2\nu+m} \overline{\widetilde{\Phi}_j^{\nu,m}(z)} \mid j, \nu > \tag{3.7}$$

where $\mathcal{N}_{\nu,m}(z)$ *is a normalization factor and* $\widetilde{\Phi}_j^{\nu,m}(.)$ *is the monopole harmonic defined in (2.15).*

Now, one of the important task to achieve is to determine explicitly the overlap relation between two GSCSs.

Proposition 3.1. *Let* $m \in \mathbb{Z}_+$ *and* $\nu \in \left\{\frac{1}{2}, 1, \frac{3}{2}, ...\right\}$. *Then, for every* $z, w \in \overline{\mathbb{C}}$, *the overlap relation between two GSCSs is given through the scalar product*

$$< w, \nu, m \mid z\ , \nu, m >_{\mathcal{H}} = \frac{(\nu+2m+1)(1+z\overline{w})^{2\nu}}{\left(\mathcal{N}_{\nu,m}(z)\mathcal{N}_{\nu,m}(w)\right)^{\frac{1}{2}}(1+z\overline{z})^{\nu}(1+w\overline{w})^{\nu}}. \tag{3.8}$$

$$\times {}_2F_1\left(-m, m+2\nu+1, 1; \frac{(z-w)(\overline{z}-\overline{w})}{(1+z\overline{z})(1+w\overline{w})}\right)$$

where ${}_2F_1$ *is a terminating Gauss hypergeometric sum.*

Proof. In view of Eq.(3.7), the scalar product of two GSCS $\mid z\ , \nu, m >$ and $\mid w\ , \nu, m >$ in $\mathcal{H}$ reads

$$< w, \nu, m \mid z\ , \nu, m >_{\mathcal{H}} = \left(\mathcal{N}_{\nu,m}(z)\mathcal{N}_{\nu,m}(w)\right)^{-\frac{1}{2}} \mathfrak{S}_{z,w}^{\nu,m} \tag{3.9}$$

where

$$\mathfrak{S}_{z,w}^{\nu,m} = \sum_{j=-m}^{2\nu+m} \widetilde{\Phi}_j^{\nu,m}(z)\overline{\widetilde{\Phi}_j^{\nu,m}(w)}. \tag{3.10}$$

Recalling Eq.(2.11), we can rewrite the finite sum (3.10) in terms of the product of the polynomial functions $Q_j^{\nu,m}(u)$ in (2.10). Next, we use the addition formula due to J. Peetre and G. Zhang ([13], p.231) involving the functions in (2.10) to obtain that

$$\mathfrak{S}_{z,w}^{\nu,m} = \frac{(2\nu+2m+1)(1+z\overline{w})^{2\nu}}{\left((1+z\overline{z})(1+w\overline{w})\right)^{\nu}} \cdot {}_2F_1\left(-m, m+2\nu+1, 1; \frac{(z-w)(\overline{z}-\overline{w})}{(1+z\overline{z})(1+w\overline{w})}\right) \tag{3.11}$$

Returning back to Eq.(3.9) and inserting the expression (3.11) we arrive at the announced result.■

Corollary 3.1. *The normalization factor in* (3.7) *is given by*

$$\mathcal{N}_{\nu,m}(z) = 2(\nu+m)+1 \tag{3.12}$$

for every $z \in \overline{\mathbb{C}}$.

Proof. We first make appeal to the relation (2.13) connecting the ${}_2F_1-$sum with the Jacobi polynomial to rewrite Eq.(3.8) as

$$\sqrt{\mathcal{N}_{\nu,m}(z)\,\mathcal{N}_{\nu,m}(w)} = \frac{(1+z\overline{z})^{-\nu}(2\nu+2m+1)(1+z\overline{w})^{2\nu}}{(1+w\overline{w})^{\nu} < w,\nu,m \mid z\ ,\nu,m >_{\mathcal{H}}} \times P_m^{(0,2\nu)}\left(1-\frac{2(z-w)(\overline{z}-\overline{w})}{(1+z\overline{z})(1+\overline{w}w)}\right). \tag{3.13}$$

The factor $\mathcal{N}_{\nu,m}(z)$ should be such that $< z,\nu,m \mid z,\nu,m >_{\mathcal{H}}= 1$. So that we put $z = w$ in (3.13) to obtain the expression

$$\mathcal{N}_{\nu,m}(z) = (2\nu+2m+1)\,P_m^{(0,2\nu)}(1)\,. \tag{3.14}$$

Finally, we apply the fact that ([14], p.57) :

$$P_n^{(\alpha,\sigma)}(1) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} \tag{3.15}$$

in the case of $\alpha = 0, n = m$ and $\sigma = 2\nu$. This ends the proof.■

Proposition 3.2. *Let* $m \in \mathbb{Z}_+$ *and* $\nu \in \left\{\frac{1}{2}, 1, \frac{3}{2}, ...\right\}$. *Then, the GSCS* $\mid z,\nu,m >$ *satisfy the following resolution of the identity*

$$\int_{\mathbb{C}} \mid z,\nu,m >< z,\nu,m \mid d\mu_{\nu,m}(z) = \mathbf{1}_{\mathcal{H}} \tag{3.16}$$

where $\mathbf{1}_{\mathcal{H}}$ *is the identity operator and* $d\mu_{\nu,m}(z)$ *is a measure which can be expressed through a Meijer's G function as*

$$d\mu_{\nu,m}(z) := (2\nu+2m+1)\,G_{11}^{11}\left(z\overline{z} \mid \begin{array}{c} -1 \\ 0 \end{array}\right) d\eta(z)\,, \tag{3.17}$$

where $d\eta(z)$ *denotes the Lebesgue measure on* $\mathbb{C}$.

Proof. We assume that the measure takes the form $d\mu_{\nu,m}(z) = \mathcal{N}_{\nu,m}(z)\,\Omega(z)\,d\eta(z)$ where $\Omega(z)$ is an auxiliary density to be determined. Let $\varphi \in \mathcal{H}$ and let us start by writing the following action

$$\mathcal{O}[\varphi] := \left(\int_{\mathbb{C}} \mid z,\nu,m >< z,\nu,m \mid d\mu_{\nu,m}(z)\right)[\varphi] \tag{3.18}$$

$$= \int_{\mathbb{C}} < \varphi \mid z,\nu,m >< z,\nu,m \mid d\mu_{\nu,m}(z) \tag{3.19}$$

Making use of Eq.(3.7) , we obtain successively

$$\mathcal{O}[\varphi] = \int_{\mathbb{C}} < \varphi \mid (\mathcal{N}_{\nu,m}(z))^{-\frac{1}{2}} \sum_{j=-m}^{2\nu+m} \overline{\widetilde{\Phi}_j^{\nu,m}(z)} \mid j,\nu >>< z,\nu,m \mid d\mu_{\nu,m}(z) \tag{3.20}$$

$$= \left(\sum_{j,k=-m}^{2\nu+m} \int_{\mathbb{C}} \overline{\widetilde{\Phi}_j^{\nu,m}(z)} \widetilde{\Phi}_k^{\nu,m}(z) \mid j,\nu >< k,\nu \mid (\mathcal{N}_{\nu,m}(z))^{-1} d\mu_{\nu,m}(z) \right) [\varphi]. \tag{3.21}$$

Replace $d\mu_{\nu,m}(z)$ by $\mathcal{N}_{\nu,m}(z)\Omega(z)d\nu(z)$, then Eq.(3.21) takes the form

$$\mathcal{O} = \sum_{j,k=-m}^{2\nu+m} \left[\int_{\mathbb{C}} \overline{\widetilde{\Phi}_j^{\nu,m}(z)} \widetilde{\Phi}_k^{\nu,m}(z) \Omega(z) d\nu(z) \right] \mid j,\nu >< k,\nu \mid. \tag{3.22}$$

Then, we need to obtain

$$\int_{\mathbb{C}} \overline{\widetilde{\Phi}_j^{\nu,m}(z)} \widetilde{\Phi}_k^{\nu,m}(z) \Omega(z) d\nu(z) = \delta_{j,k}. \tag{3.23}$$

For this we recall the orthogonality relation satisfied by monopole harmonics in (2.15) in $L^2\left(\overline{\mathbb{C}}, (1+z\overline{z})^{-2} d\eta(z)\right)$, as

$$\int_{\overline{\mathbb{C}}} \overline{\widetilde{\Phi}_j^{\nu,m}(z)} \widetilde{\Phi}_k^{\nu,m}(z) (1+z\overline{z})^{-2} d\eta(z) = \delta_{j,k}. \tag{3.24}$$

This suggests us to set $\Omega(z) := (1+z\overline{z})^{-2} d\eta(z)$. By making us of the identity ([19]) :

$$G_{11}^{11}\left(u \mid \begin{array}{c} a \\ b \end{array} \right) = \Gamma(1-a+b) u^b (1+u)^{a-b-1} \tag{3.25}$$

for $u = z\overline{z}$, $a = -1$ and $b = 0$, we arrive at the announced form for the measure $d\mu_{\nu,m}$ in (3.17). Because of this this measure, Eq.(3.22) reduces to

$$\mathcal{O} = \sum_{j,k=-m}^{2\nu+m} \delta_{j,k} \mid j,\nu >< k,\nu \mid = \mathbf{1}_{\mathcal{H}}. \tag{3.26}$$

This ends the proof.■

Remark 3.1. Note that when $m = 0$, Eq.(3.16) leads to Eq.(3.3). For $m \neq 0$, the fact that we have written the measure $d\mu_{\nu,m}(z)$ in (3.17) in terms of the Meijer's G-function could be of help when tackling the "*photon-added coherent states* ($PACS$)" problem for the GSCS under consideration.

Proposition 3.3. *The states* $\mid z,\nu,m>$ *satisfy the continuity property with respect to the labelling point* z. *That is, the norm of the difference of two states*

$$\rho_{\nu,m}(z,w) := \| \mid z,\nu,m> - \mid w,\nu,m> \|_{\mathcal{H}} \tag{3.28}$$

goes to zero whenever $z \to w$.

Proof. By using the fact that any GSCS is normalized by the factor given in (3.12), a direct calculation enables us to write the square of the quantity in (3.28) as

$$\left(\rho_{\nu,m}(z,w) \right)^2 = 2\left(1 - \operatorname{Re} < z,\nu,m \mid w,\nu,m> \right). \tag{3.29}$$

So it is clear that when $z \to w$, the terminating Gauss hypergeometric function goes to 1 and the prefactor in (3.8) goes to $(2\nu + 2m + 1)$. Therefore,the overlap takes the value 1 and consequently $\rho_{\nu,m}(z,w) \to 0$ in (3.29).■

As we can see, these GSCS are independent of the basis $\mid j, \nu >$ we use and the only condition which is implicitly fulfilled is the orthonormality of the basis vectors of $\mathcal{H}$. But if we want to attach our GSCS to a concrete quantum system then a Hamiltonian operator should be specified together with a corresponding explicit eigenstates basis. This will be the goal of the next section.

4 The Kravchuk oscillator

The Kravchuk polynomials $K_k^{(p)}(x,N)$ of degree $k = 0,1,2,...,N$, in the variable $x \in [0,N]$ and of the parameter $0 < p < 1$, are related to the binomial probability distribution [5]. They satisfy the well known three-term recurrence relation :

$$(x - k - p(N-2k)) K_k^{(p)}(x,N) \tag{4.1}$$

$$= (k+1) K_{k+1}^{(p)}(x,N) + p(1-p)(N-k+1) K_{k-1}^{(p)}(x,N)$$

and can be defined in terms of the Gauss hypergeometric function through

$$K_k^{(p)}(x,N) := (-1)^k p^k \begin{pmatrix} N \\ k \end{pmatrix} . {}_2F_1\left(-k,-x,-N;\frac{1}{p}\right). \tag{4.2}$$

For each fixed nonzero positive integer N, the $N+1$ Kravchuk polynomials $\left\{K_k^{(p)}(x,N)\right\}_{k=0}^{N}$ are an orthogonal set with respect to a discrete weight function with finite support, namely

$$\sum_{j=0}^{N} \varrho(j) K_k^{(p)}(j,N) K_n^{(p)}(j,N) = \frac{N!}{k!(N-k)!}(p(1-p))^n \delta_{k,n} \tag{4.3}$$

and the binomial weight function

$$\varrho(x) = \frac{N!}{\Gamma(x+1)\Gamma(N-x+1)} p^x (1-p)^{N-x}. \tag{4.4}$$

The Kravchuk functions can be defined as the ket vectors with wavefunctions

$$\phi_k^{(p)}(x,N) := (d_k)^{-1} \sqrt{\varrho(x+Np)} K_k^{(p)}(Np+x,N) \tag{4.5}$$

where $d_k^2 = \frac{N!}{k!(N-k)!}(p(1-p))^k$, $k \in \{0,1,...,N\}$ and $-Np \le x \le (1-p)N$. They obey the following discrete orthogonality relation

$$\sum_{j=0}^{N} \phi_k^{(p)}(x_j,N) \phi_n^{(p)}(x_j,N) = \delta_{n,k}. \tag{4.6}$$

at the points $x_j = (j - pN)$. Following ([8], p.370), the functions $\phi_k^{(p)}(x, N)$ are eigenfunctions of the Kravchuk oscillator with the Hamiltonian

$$H^N := 2p(1-p)N + \frac{1}{2} + (1-2p)\frac{\xi}{h} - \sqrt{p(1-p)}\left(\alpha(\xi)e^{h\partial_\xi} + \alpha(\xi - h)e^{-h\partial_\xi}\right) \tag{4.8}$$

where

$$h = \sqrt{2Np(1-p)}, \alpha(\xi) = \sqrt{((1-p)N - h^{-1}\xi)(pN + 1 + h^{-1}\xi)}. \tag{4.9}$$

This operator is acting in the Hilbert space $l^2(\xi)$ with orthonormal basis consisting of Kravchuk functions (4.5) which verify

$$H\phi_k^{(p)}(x, N) = \left(k + \frac{1}{2}\right)\phi_k^{(p)}(x, N), \; k = 0, 1, ..., N. \tag{4.10}$$

It have been also pointed out [8] that these functions coincide with the harmonic oscillator functions in the limit as $N \to \infty$, namely

$$\lim_{N\to\infty} h^{-\frac{1}{2}}\phi_k^{(p)}\left(h^{-1}\xi, N\right) = \left(\sqrt{\pi}2^k k!\right)^{-\frac{1}{2}} H_k(\xi)\, e^{-\frac{1}{2}\xi^2} \tag{4.11}$$

where $H_k(.)$ are the Hermite polynomials; see also ([20], p.133).

Finally, in the subsequent we will use the notation $q = 1 - p$ and we will be concerned with the Kravchuk functions

$$\phi_k^{(p,q)}(x, N) := K_k^{(p)}(x + Np, N)\sqrt{\frac{k!(N-k)!p^{Np+x}q^{Nq-x}}{p^k q^k \Gamma(Np + x + 1)\Gamma(Nq - x + 1)}} \tag{4.12}$$

Remark.4.1. The normalized Kravchuk function in (4.12) can also be expressed in terms of the Wigner **d**-function by $(-1)^{s-r}\mathbf{d}_{s,r}^j(\beta) \equiv \phi_k^{(p,q)}(x, N)$ where $j = N/2$, $k = j - s$, $x = j - r$ and $p = \sin^2(\beta/2)$; see $[21] - [22]$. Note also that the group theoretical interpretation of the dynamical $su(2)$ algebra for the Kravchuk functions can be found in [23]. For more details we refer to [24] and references therein.

Remark 4.2. For $p = 1/2$ it is interesting [23] that in this situation the double commutator of the Hamiltonian (4.8) with the variable x satisfies the equation $\left[H^N(x), \left[H^N(x), x\right]\right] = x$ which can be viewed as the difference analogue of the equation of motion of the linear harmonic oscillator in the Schrödinger representation as pointed out in ([8], p.371).

5 Generalized spin coherent states for the Kravchuk oscillator

We now define a class of generalized spin coherent states (GSCS) for the Kravchuk oscillator according to definition (3.1) as follows.

Definition 5.1. *For $m \in \mathbb{Z}_+$, $2\nu = 1, 2, ...,$ and $0 < p < 1$ with $q = 1 - p$. A class of GSCS for the Kravchuk oscillator are defined by*

$$| z, \nu, m >_{(p,q)} := (\mathcal{N}_{\nu,m}(z))^{-\frac{1}{2}} \sum_{k=0}^{2(\nu+m)} \overline{\widetilde{\Phi}_k^{(\nu,m)}(z)} \mid \phi_k^{(p,q)}(\bullet, 2(\nu+m)) > \quad (5.1)$$

where $\mathcal{N}_{\nu,m}(z)$ is the factor in (3.12), $\widetilde{\Phi}_k^{(\nu,m)}(z)$ are the monopole harmonics defined in (2.15) and $\phi_k^{(p,q)}(\bullet, 2(\nu+m))$ are the Kravchuk functions (4.12) with $N = 2(\nu + m)$.

Proposition 5.1. *The wavefunction of these GSCS in (5.1) are of the form*

$$< \quad x \mid z, \nu, m >_{(p,q)} = \frac{(2\nu + 2m)!}{\overline{z}^m (1 + z\overline{z})^\nu} \sqrt{\frac{(2\nu + m)!m!p^{2(\nu+m)p+x} q^{2(\nu+m)q-x}}{\Gamma(2(\nu+m)p + x + 1)\Gamma(2(\nu+m)q - x + 1)}} \quad (5.2)$$
$$\times \sum_{k=0}^{2(\nu+m)} \frac{(-1)^k \overline{z}^k P_k^{(-2(\nu+m)-1, -x+2(\nu+m)q-k)}\left(1 - \frac{2}{p}\right)}{(-2(\nu+m))_k (2(\nu+m) - k)!} \sqrt{\frac{p^k}{q^k}} P_m^{(k-m, 2\nu+m-k)}\left(\frac{1 - z\overline{z}}{1 + z\overline{z}}\right)$$

Proof. We start from Eq.(5.1) by writing

$$< x \mid z, \nu, m >_{(p,q)} = (N+1)^{-\frac{1}{2}} \sum_{k=0}^{N} \overline{\widetilde{\Phi}_k^{(\nu,m)}(z)} \phi_k^{(p,q)}(x, N) \quad (5.3)$$

where the monopole harmonic function

$$\widetilde{\Phi}_k^{(\nu,m)}(z) = \sqrt{\frac{(N+1)(2\nu+m)!m!}{k!(N-k)!}} \frac{z^{k-m}}{(1 + z\overline{z})^\nu} P_m^{(k-m, N-m-k)}(u) \quad (5.4)$$

is obtained from the expression $\widetilde{\Phi}_j^{(\nu,m)}(z)$, $-m \leq j \leq 2\nu + m$ by setting $k = m + j$ and $u = (1 - z\overline{z})(1 + z\overline{z})^{-1}$. For $k \in \{0, 1, 2, ..., N\}$ and $-\frac{N}{2} \leq x \leq \frac{N}{2}$. If the Kravchuk polynomial is expressed in terms of the ${}_2F_1$-sum by using (4.2), then we can rewrite the functions $\phi_k^{(p,q)}(.)$ as

$$\begin{aligned} \phi_k^{(p,q)}(x, N) \quad &= \quad \frac{(-1)^k N!}{\sqrt{k!(N-k)!}} \sqrt{\frac{p^k q^{-k} p^{Np+x} q^{Nq-x}}{\Gamma(Np + x + 1)\Gamma(Nq - x + 1)}} \quad (5.5) \\ &\cdot {}_2F_1\left(-k, -(x + Np), -N; \frac{1}{p}\right). \end{aligned}$$

We make appeal to the relation (2.13) in order to write the ${}_2F_1$-sum in terms of the Jacobi polynomial as

$${}_2F_1\left(-k, -(x + Np), -N; \frac{1}{p}\right) = \frac{k!}{(-N)_k} P_k^{(-N-1, -x-k+Nq)}\left(1 - \frac{2}{p}\right). \quad (5.6)$$

Therefor, Eq.(5.5) takes the form

$$\phi_k^{(p,q)}(x,N) = \frac{(-1)^k N!}{\sqrt{(N-k)!}}\sqrt{\frac{p^k q^{-k} k! p^{Np+x} q^{Nq-x}}{\Gamma(Np+x+1)\Gamma(Nq-x+1)}} \times \frac{1}{(-N)_k} P_k^{(-N-1,-x-k+Nq)}\left(1-\frac{2}{p}\right). \tag{5.7}$$

Returning back to (5.3), we get successively

$$\frac{1}{\sqrt{N+1}}\sum_{k=0}^{N}\overline{\widetilde{\Phi}_k^{(\nu,m)}(z)}\phi_k(x,N) = \sum_{k=0}^{N}\sqrt{\frac{(2\nu+m)!m!}{k!(N-k)!}}\frac{\overline{z}^{k-m}}{(1+z\overline{z})^{\nu}}P_m^{(k-m,N-m-k)}(u) \tag{5.8}$$

$$\times\frac{(-1)^k N!}{\sqrt{(N-k)!}}\sqrt{\frac{k! p^{Np+x+k} q^{Nq-x-k}}{\Gamma(Np+x+1)\Gamma(Nq-x+1)}}\frac{1}{(-N)_k}P_k^{(-N-1,-x-k+Nq)}\left(1-\frac{2}{p}\right)$$

$$= \frac{N!\overline{z}^{-m}}{(1+z\overline{z})^{\nu}}\sqrt{\frac{(2\nu+m)!m!p^{Np+x}(1-p)^{Nq-x}}{\Gamma(Np+x+1)\Gamma(Nq-x+1)}} \tag{5.9}$$

$$\times\sum_{k=0}^{N}\frac{(-1)^k\overline{z}^k}{(-N)_k(N-k)!}\sqrt{\frac{p^k}{q^k}}P_m^{(k-m,N-m-k)}(u)\,P_k^{(-N-1,-x+Nq-k)}\left(1-\frac{2}{p}\right).$$

By Eq.(5.9) we arrive at the expression (5.2). ■

Now, keeping $N = 2(\nu+m)$ and denoting by $\mathcal{H}$ the $(2N+1)-$dimensional Hilbert space generated by the Kravchuk eigenstates then we can construct "*à la Bargmann*" [25] for any state vector $\mid\phi>$ in $\mathcal{H}$ the corresponding function in the eigenspace $\mathcal{A}_m^{\nu}(\mathbb{S}^2)$ defined in (2.12). This function is the Bargmann transform of the state $\mid\phi>$, say $\mathcal{B}_{\nu,m}[\phi]$, which is performed by applying the coherent state transform formalism [4]. Precisely, for each $m\in\mathbb{Z}_+$, it is defined as $\mathcal{B}_{\nu,m}:\mathcal{H}\to\mathcal{A}_m^{\nu}(\mathbb{S}^2)$ by

$$\mathcal{B}_{\nu,m}[\phi](z) := (2(\nu+m)+1)^{\frac{1}{2}}\langle\phi\mid z,\nu,m\rangle_{\mathcal{H}}. \tag{5.10}$$

Thus, Eq.(5.10) together with Eq.(3.16) lead to the following representation of any state vector $\mid\phi>$ in $\mathcal{H}$ in terms of the constructed GSCS $\mid z,\nu,m>$ as

$$\mid\phi>=\int_{\mathbb{C}}d\mu_{\nu,m}(z)\,(2(\nu+m)+1)^{-\frac{1}{2}}\mathcal{B}_{\nu,m}[\phi](z)\mid z,\nu,m>. \tag{5.11}$$

Finally, taking into account Eq.(3.17), we obtain from (5.11) the equality

$$\langle\phi\mid\phi\rangle_{\mathcal{H}} = \int_{\mathbb{C}}|\mathcal{B}_{\nu,m}[\phi](z)|^2 G_{11}^{11}\left(z\overline{z}\mid\begin{array}{c}-1\\0\end{array}\right)d\eta(z), \tag{5.12}$$

where the Meijer's G-function and the Lebesgue measure $d\eta(z)$ are employed. These notation could be of help when tackling the photon-added coherent states problem for the constructed GSCS $\mid z,\nu,m>$ as mentioned in a previous remark.

Remark 3.1. We should note that a set of coherent states attached to spherical Landau levels, which form is similar to (5.1), have been performed in [6] and $[11]-[26]$ with the choice for the Hilbert space carrying them as the space of polynomials of degree less than $2\nu+2m+1$ endowed with an orthonormal basis of the form: $\phi_j(\varkappa)=\left((2\nu+2m!)\left((2\nu+m-j)!\right)^{-1}\left((j+m)!\right)^{-1}\right)^{\frac{1}{2}}\varkappa^{j+m}$ where $\varkappa\in\mathbb{C}$ and $0\leq j\leq 2(\nu+m)$.

6 The case m=0

Now, from the above proposition 5.1 , we can deduce the following result.

Corollary 6.1. *For $2\nu=1,2,...,$ and $0<p<1$ with $q=1-p$. The wave functions for GSCS in (5.2), corresponding to the lowest spherical Landau level* $\lambda_{\nu,0}$ in *(2.5) are of the form*

$$< \ x\mid z,\nu,0>_{(p,q)}=\frac{\sqrt{N!}}{(1+z\overline{z})^{\nu}}\sqrt{\frac{p^{Np+x}q^{Nq-x}}{\Gamma(Np+x+1)\Gamma(Nq-x+1)}}\times\left(1+\sqrt{\frac{q}{p}}\overline{z}\right)^{x+Np}\left(1-\sqrt{\frac{p}{q}}\overline{z}\right)^{Nq-x} \quad (6.1)$$

where $N=2\nu$.

Proof. We start by putting $m=0$ in the expression (5.3). That is

$$< x\mid z,\nu,0>_{(p,q)}=(N+1)^{-\frac{1}{2}}\sum_{k=0}^{N}\overline{\widetilde{\Phi}_k^{(\nu,0)}(z)}\phi_k^{(p,q)}(x,N). \quad (6.2)$$

Using the fact that $P_0^{(\alpha,\beta)}(u)=1$, then Eq.(5.4) reduces to

$$\widetilde{\Phi}_k^{(\nu,0)}(z)=\sqrt{\frac{(N+1)N!}{k!(N-k)!}}\frac{z^k}{(1+z\overline{z})^{\nu}}. \quad (6.3)$$

Replacing the expression (6.3) in Eq.(6.2), we obtain that

$$< x\mid z,\nu,0>_{(p,q)}=\frac{\sqrt{N!}}{(1+z\overline{z})^{\nu}}\sum_{k=0}^{N}\sqrt{\frac{1}{k!(N-k)!}}\overline{z}^k\phi_k^{(p,q)}(x,N). \quad (6.4)$$

On the other hand, we make use of Eq.(5.5), to rewrite (6.4) as

$$< x\mid z,\nu,0>_{(p,q)}=\frac{\sqrt{N!}}{(1+z\overline{z})^{\nu}}\sqrt{\frac{p^{Np+x}q^{Nq-x}}{\Gamma(Np+x+1)\Gamma(Nq-x+1)}} \quad (6.5)$$

$$\times\sum_{k=0}^{N}\left(-\overline{z}\sqrt{\frac{p}{q}}\right)^k.\begin{pmatrix}N\\k\end{pmatrix}.{}_2F_1\left(-k,-(x+Np),-N;\frac{1}{p}\right).$$

Now, with the help of the generating function $([27], p.184)$:

$$\left(1-\frac{q}{p}t\right)^{\zeta}(1+t)^{N-\zeta}=\sum_{n=0}^{N}\left(\begin{array}{c}N\\k\end{array}\right)K_{k}(\zeta,p,N)\,t^{n}. \tag{6.6}$$

where $\zeta=0,1,2,...,N$ and $K_{k}(x,p,N)={}_{2}F_{1}\left(-k,-x,-N;\frac{1}{p}\right)$. We apply it for $t=-\overline{z}\sqrt{\frac{p}{q}}$ and $\zeta=x+Np$ to find that

$$\sum_{k=0}^{N}\left(-\overline{z}\sqrt{\frac{p}{q}}\right)^{k}\left(\begin{array}{c}N\\k\end{array}\right).{}_{2}F_{1}\left(-k,-(x+Np),-N;\frac{1}{p}\right) \tag{6.7}$$
$$=\left(1+\sqrt{\frac{q}{p}}\overline{z}\right)^{x+Np}\left(1-\sqrt{\frac{p}{q}}\overline{z}\right)^{Nq-x}.$$

Finally, in view of (6.7), Eq.(6.5) takes the form

$$<x\mid z,\nu,0>_{(p,q)}=\frac{\sqrt{N!}}{(1+z\overline{z})^{\nu}}\sqrt{\frac{p^{Np+x}q^{Nq-x}}{\Gamma(Np+x+1)\Gamma(Nq-x+1)}}\left(1+\sqrt{\frac{q}{p}}\overline{z}\right)^{Np+x}\left(1-\sqrt{\frac{p}{q}}\overline{z}\right)^{Nq-x} \tag{6.8}$$

This ends the proof of the corollary.■

We should note that in [9] Chenaghlou and Faizy have constructed a class of Klauder-Perelomov coherent states by acting on the ground state function

$$\psi_{0}(y)=\sqrt{\frac{N!p^{y}(1-p)^{N-y}}{y!(N-y)!}},0<p<1 \tag{6.9}$$

via a displacement operator defined by two generators of the Lie algebra $so(3)$. The wave functions of their coherent states is were of the form($[9]$, Eq.(38)):

$$<y\mid z,N>^{KP}:=\left(1+\frac{p}{1-p}z\overline{z}\right)^{-\frac{1}{2}N}(1+z)^{y}\left(1-\frac{pz}{1-p}\right)^{N-y}\sqrt{\frac{N!p^{y}(1-p)^{N-y}}{y!(N-y)!}}. \tag{6.10}$$

To establish a connection between the coherent states $\mid z,N>^{KP}$in (6.10) and our constructed coherent states in the case $m=0$ we need to make a little change of variables in Eq.(6.1). We precisely consider the following replacements: $x\to y-Np$ and $z\to\sqrt{\frac{p}{q}}\overline{z}$. By this way, one can easily check that the following fact:

$$<y-Np\mid\sqrt{\frac{p}{q}}\overline{z},\nu,0>_{(p,q)}=<y\mid z,N>^{KP}. \tag{6.11}$$

Finally, if one particularize the unitary transform (5.10) for the case $m=0$ then one can recover the *analytic* coherent states representation of the any state vector $\mid\phi>$ in $\mathcal{H}$ as discussed in $[9]$.

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